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We discuss a model in which a quantum particle passes through δ potentials arranged in an increasingly sparse way. For infinitely many barriers we derive conditions, expressed in terms ergodic properties of wave function phases, which ensure that the point and absolutely continuous parts are absent leaving a purely singularly continuous spectrum. For a finite number of barriers, the transmission coefficient shows extreme sensitivity to the particle momentum with fluctuation in many different scales. We discuss a potential application of this behavior for erasing the information carried by the wave function.

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The prospective use of quantum states as information carriers put new emphasis on tailoring systems which could process the wave function in a desired way. One of such devices, potentially of crucial importance, is an information eraser, or briefly *shredder*. If we look for ways in which a quantum system could achieve this goal, a natural idea is to seek situations where a slight change of the input is magnified significantly in the output.

In classical mechanics, the sensitive dependence on the initial condition of that kind is the essence of the deterministic randomness exhibited by non-integrable dynamical systems [1]. In quantum mechanics, on the other hand, such sensitive dependence has been an elusive object despite many efforts, and a widely accepted counterpart to the mentioned classical behavior is missing. Instead, “quantum signatures” of chaos are often studied which typically manifest themselves through statistical properties of spectra of various observables [2]. There has been several attempts to find sensitive dependence in quantum mechanics using time periodic models [3,4]. However, they are not easy to solve and the numerical results often contain ambiguities. In this letter, we abandon such traditional schemes, and instead propose a simple one-dimensional model with a potential which has many different length scales.

We start with an infinite number of barriers whose distances grow, and ask for conditions under which its spectrum exhibits irregularity, defined in mathematical terms as being singularly continuous. We will have unexpected encounter with an associated classical system whose ergodicity brings about the appearence of purely singularly continuous spectrum. We then consider a scattering system in which the number of barriers is finite. Through numerical analysis, it is shown that the transmission coefficient exhibits a highly irregular behavior. We also show that a wave packet passing through the system undergoes randomizing during which the shape information contained in the input wave function gets lost.

We choose an explicitly solvable model of a particle on the line interacting with $(2N+1)$ δ potentials having the same coupling constant, which is described by the stationary Schrödinger equation

$$\left[-\frac{d^2}{dx^2}+\sum_{n=-N}^{N} v\delta(x-x_n)\right]\varphi(x)=k^2\varphi(x), \tag{1}$$

where the positions x_i are indexed in ascending order. Properties of such systems depend crucially on the positioning of the δ interactions. In our case they are arranged symmetrically, $x_{-n}=-x_n$ with $x_0=0$, and in such a way that their positions are increasingly sparse;

$$x_{|n|+2}-x_{|n|+1}>x_{|n|+1}-x_{|n|}\,. \tag{2}$$

We are particularly interested in the situations where the distances grow polynomially, exponentially, or faster,

$$x_{|n|}=b\,|n|^{\beta}+c\,\mathrm{e}^{a|n|^{\gamma}} \tag{3}$$

for $n\neq 0$, with suitable positive a, b, c, and $\beta>1$, $\gamma\geq 1$.

We employ transfer matrices which relate solutions of Eq. (1) at different points. We define the two component vector $\Phi(x)$ from the wave function and its derivative as

$$\Phi(x)=\begin{pmatrix}\varphi(x)\\ \varphi'(x)\end{pmatrix}; \tag{4}$$

then the transfer matrix $\mathcal{M}(x,y)$ is given by

$$\Phi(x)=\mathcal{M}(x,y)\Phi(y). \tag{5}$$

Because of the zero-range nature of the interaction [5], the transfer matrix $\mathcal{M}(x,y)$ is expressed as interlaced products of ones describing the free motion with momentum k,

$$\mathcal{M}_k^{\rm free}(x,y)=\begin{pmatrix}\cos k(x-y) & \frac{1}{k}\sin k(x-y)\\ -k\sin k(x-y) & \cos k(x-y)\end{pmatrix}, \tag{6}$$

and the point interaction transfer matrices

$$\mathcal{M}(x_n+0,x_n-0) \equiv \mathcal{M}_v = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}. \tag{7}$$

We do not index the latter since the δ potentials are supposed to be identical. If $x > y$ the component matrices of $\mathcal{M}(x,y)$ are multiplied in the order reversed to that of the involved intervals.

To express the amplitudes for the left-to-right scattering over a finite number of barriers, we suppose

$$\begin{aligned} \varphi(x) &= \mathrm{e}^{ikx} + r(k)\mathrm{e}^{-ikx} \quad && x < x_{-N} \\ &= \qquad t(k)\mathrm{e}^{ikx} && x > x_N \end{aligned} \tag{8}$$

and match the boundary values at $x_{\pm N}$ using the matrix $\mathcal{M} \equiv \mathcal{M}(x_N+0, x_{-N}-0)$; then an easy computation using $\det \mathcal{M} = 1$ gives, in particular,

$$t(k) = \frac{2ik}{\mathcal{M}_{21} - ik(\mathcal{M}_{22}+\mathcal{M}_{11}) - k^2\mathcal{M}_{12}}\,. \tag{9}$$

Before making use of the last formula, let us look what happens in the limit $N \to \infty$ when the scattering loses meaning; we focus on the change of the spectral characteristics of the system. Traditionally, most attention has been paid in physics literature to two types of spectra the point and the absolutely continuous. Recently, there has been a growing recognition of the importance of the third type, which is mathematically well studied, the singularly continuous spectrum. Here, we ask ourselves under which condition the spectrum of our system is purely singularly continuous. In view of the symmetry it is sufficient to consider the operator

$$H_\vartheta = -\frac{d^2}{dx^2} + \sum_{n=1}^{\infty} v\delta(x-x_n) \tag{10}$$

on $L^2(0,\infty)$ with the boundary condition

$$\varphi(0)\cos\vartheta + \varphi'(0)\sin\vartheta = 0\,, \tag{11}$$

in particular, our case with the δ potential of strength v at the origin corresponds to $\vartheta = -\arcsin(1+\frac{v^2}{4})^{-1/2}$.

Several recent papers investigated the relations between the spectrum of the operator (10) and growth properties of the corresponding transfer matrices. A sufficient condition for the absence of the point spectrum is [6]

$$\int_0^\infty \frac{dx}{\|\mathcal{M}(x,0)\|^2} = \infty\,. \tag{12}$$

The norm of the transfer matrix can be estimated by the product of norms of its component matrices,

$$\|\mathcal{M}(x_n,0)\| \le \nu^2 \left(\max(k,k^{-1})\right)^n \equiv \mathcal{M}[v,k]^n, \tag{13}$$

with

$$\nu := \frac{|v|}{2} + \sqrt{\frac{v^2}{4}+1} \tag{14}$$

Denoting $\Delta_n := x_n - x_{n-1}$, we have

$$\int_0^{x_N} \frac{dx}{\|\mathcal{M}(x,0)\|^2} \ge \sum_{n=1}^{N} \frac{\Delta_n}{\|\mathcal{M}[v,k]\|^{2n}}\,, \tag{15}$$

so the integral diverges, for example, if one has $\Delta_n \ge c_1 n^{-1/2}\,\mathrm{e}^{an}$ for some $c_1 > 0$ and

$$a > 2\left\{\ln \max(k,k^{-1}) + \ln\nu\right\}. \tag{16}$$

If we want a condition independent of the parameters we have to require a slightly faster growth, for instance, that of (3) with any $\gamma > 1$.

Sparse potentials with growing Δ_n are known to be good candidates to produce singularly continuous spectra. Already the first example by Pearson [7] is of this kind, many others can be found in [6,8]. In distinction to a typical Pearson potential, however, the height of the barriers does not decrease here. This makes it more difficult to exclude the absolutely continuous spectrum. To show where is the core of the problem we employ the modified Prüfer technique [8]. It is based on the Ansatz

$$\varphi(x) = R(x)\sin\theta(x)\,, \quad \varphi'(x) = kR(x)\cos\theta(x)\,. \tag{17}$$

The amplitude $R(x)$ is constant in the absence of a potential, so the solution of the Schrödinger equation in the interval (x_{n-1}, x_n) between the barriers is

$$\varphi(x) = R_n \sin\left[\theta_n + k(x-x_{n-1})\right]. \tag{18}$$

The δ coupling at the point x_n means the wave function continuity together with

$$\varphi'(x_n+0) - \varphi'(x_n-0) = v\varphi(x_n)\,. \tag{19}$$

This yields a system of equations for R_n, R_{n+1} which is solvable under the Eggarter condition [9,10]

$$\cot\theta_{n+1} = \cot\left[\theta_n + k\Delta_n\right] + \frac{v}{k}\,. \tag{20}$$

This determines the phase of the wave function in an iterative way. For the ratio of the amplitudes $r_n := R_{n+1}/R_n$, we get

$$r_n^2 = \frac{\sin^2\left[\theta_n + k\Delta_n\right]}{\sin^2\theta_{n+1}} = \mu + \sqrt{\mu^2-1}\,\sin\beta_n \tag{21}$$

with the definitions

$$\beta_n := 2\theta_n + 2k\Delta_n - \alpha, \tag{22}$$

and

$$\mu := 1 + \frac{v^2}{2k^2}, \quad \alpha := \arccos\left(1+\frac{v^2}{4k^2}\right)^{-1/2}. \tag{23}$$

The new phase variable β_n is determined by the evolution

$$\cot\left(\frac{\beta_{n+1}+\alpha}{2} - k\Delta_{n+1}\right) = \cot\left(\frac{\beta_n+\alpha}{2}\right) + \frac{v}{k}. \quad (24)$$

By Theorem 1.1 of [8], the absolutely continuous spectrum will be absent if $\|\mathcal{M}(x,0)\|$ tends to infinity at least for some sequence of points and *a.e.* k. Since

$$\|\Phi(x)\|^2 \geq \min(1,k)R(x)^2, \quad (25)$$

it is enough to demonstrate the growth for a subsequence of $\{R_n\}$, or equivalently, that the quantity

$$\sum_{n=1}^{N} \ln r_n^2 \quad (26)$$

is exploding for a subsequence of the indices N. Now, for a continuous function $F \in C([-\pi,\pi])$, one has

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} F(\beta_n) = \int_{-\pi}^{\pi} F(\beta)\, d\beta \quad (27)$$

if the sequence $\{\beta_n\}_{n=1}^{\infty}$ is *uniformly distributed* in $[-\pi,\pi]$ [11]. In our case, we have $F(\beta_n) := \ln r_n^2$. Using the fact that the sine is an odd function we find

$$\int_{-\pi}^{\pi} F(\beta)\, d\beta = \int_0^{\pi} \ln\left(\mu^2\cos^2\beta + \sin^2\beta\right)\, d\beta > 0 \quad (28)$$

for any $\mu > 1$. It follows that the sequence $\{R_N\}$ is exponentially growing if $\{\beta_n\}$ is uniformly distributed. Thus the problem of singularly continuous quantum spectrum is turned into that of dynamical properties of the *classical map* (24).

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FIG. 1. (a) The histogram shows the distribution of the phase β_n in the interval $[-\pi,\pi)$ for $v = k = 1$ after $N = 20000$ iterations. (b) The distribution of θ_n for the same values.

We shall not attempt to prove this property and limit ourselves to illustrating it for $x_n = n^2$. The results are shown in Fig. 1; similar behavior may be observed for other values v, almost all [12] values of the momentum k, as well as for $x_n = n^\alpha$ with $\alpha > 1$. Although the original phases θ_n may not be uniformly distributed as shown in the Fig. 1, it is sufficient that $\{\theta_n\}$ is ergodic, if the sequence $\{2k\Delta_n\}$ is uniformly distributed, which turns out to be the case when k is an irrational multiple of π, for example, for $x_n = n^2$ [11].

Let us sort out the above discussion. If the number of barriers is infinite the absolute continuity for Hamiltonians of the type (1) requires a particular arrangement, e.g., a periodic one. On the other hand, randomly distributed δ barriers yield almost surely a pure point spectrum by Anderson localization [13]. The exclusive character of absolutely continuous spectra makes easier to get rid of them; the above results suggests that a power-like sparseness may be sufficient. At the same time, the borderline between dense point and singularly continuous spectra is extremely unstable and an exponential or faster growth may be needed to get a purely singularly continuous spectrum.

Let us return to systems with a finite N. When the number of barrriers is finite, the positive energy spectra of the system is purely absolutely continuous. However, one expects certain trace of singular behavior in observable quantities even with finite N. One suspects, for example, that the poles of the scattering matrix is distributed differently for sparse potentials, which should be reflected in the increasingly wild behavior of scattering amplitudes as one increase the sparseness of the potential with fixed finite N.

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FIG. 2. Transmission probability and transmission amplitude over $2N + 1 = 11$ barriers for different sparseness.

That expectation is confirmed in Figure 2, in which we plot the transmission coefficient $|t(k)^2|$ as a function of incident momentum k. The number of barriers is chosen to be $2N + 1 = 11$. One has increasing sparseness from top to bottom. Irregular fluctuation of the transmission coefficient which is reminiscent of the *classical* chaotic scattering [1] is clearly observed for sparse potentials. Note also the "ergodic" behavior of the trajectory of $\{\mathrm{Re}t(k), \mathrm{Im}t(k)\}$ with increasing sparseness. In Figure 3, we plot the transmission coefficient for a larger number of barriers $2N + 1 = 21$ with exponential sparseness $x_n = e^n$. The top graph is magnified by factor of 10 in the second graph, part of which is magnified again by factor 10 in the third graph. From this, one can discern the fact that the irregular fluctuation of quantum scattering matrix takes place in many diffrent scale. Notice that the despite the irregularity the systems of sparse barriers discussed here still exhibit a considerable transmission: for the same number of *randomly positioned* barriers Anderson localization takes place and $|t(k)|^2$ is practically identical to zero.

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FIG. 3. Transmission amplitude $|t(k)|^2$ for exponentially sparse 21 barriers in diffrent k scale.

In actual experimental settings, it is often difficult to prepare the particle beam of single wavelength. One would rather consider the problem as the transmission of *wave packet*. When one has the incoming wave with Fourier component $\varphi(k)$, the outcoming wave will have the spectra $t(k)\varphi(k)$. The initial wave packet $\varphi_{in}(x,t)$ will be proccessed after the transmission to give the outcomming wave packet

$$\varphi_{out}(x,t) = \int_{-\infty}^{\infty} t(x-y)\varphi_{in}(y,t)\,dy, \tag{29}$$

where $t(x)$ is the Fourier transform of $t(k)$. Because of the fluctuation in all scale is present in $t(k)$, the quantity $t(x)$ has a slow decay at large distances. As a consequence, any information contained in the shape of the wave packet $\varphi_{in}(x,t)$ is effectively washed out in $\varphi_{out}(x,t)$ after the convolution (29).

In summary, we have analyzed transport of a quantum particle through a sparse family of δ barriers of identical strength. If the barriers are arranged in a non-random way there is a non-negligible transmission. When the barriers are placed sparsely, the transmission coefficient shows an irregular dependence on incoming momentum in self-similar manner, and the shape of wave packets passing through the system is randomized. Properties of the transport can be studied through spectral properties of infinite barrier systems. Because of the solvability of the problem, a rather detailed analysis can be carried out, and a hidden relation between the spectral properties of the system and the ergidicity of a certain classical dynamical system is revealed. We stress that the simple model considered here is not so far removed from experimentally realizable system with existing technology. Thus we conclude that the realization of initial-condition sensitive, information erasing process in purely quantal setting may be already within our reach.

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